A NONLINEAR MICROECONOMIC MODEL OF GOODS PRODUCTION AND SALE USING FUNCTIONAL ANALYSIS

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ABSTRACT
Mathematical methods have been used with increasing success in solving particular tasks of dynamic process modelling, both in macroeconomics and in microeconomics. The efforts to simulate processes in situations when experiments on real objects cannot be performed or such experiments would be uneconomical are the most frequent reason for modelling in microeconomics. This paper presents a nonlinear dynamic model of production, storage and sale of daily consumer goods, ac-counting for the impact of history on the behaviour of the respective values at present. The model is described using the so-called ordinary delay differential equations and may be used in production planning or as a part of models addressing issues of cooperation or competition. The solution is demonstrated on a practical example and is presented graphically.

KEYWORDS:*goods, delay differential equations, functional differential equation, microeconomic model*

JEL CLASSIFICATION: *C02, C69*

1. INTRODUCTION

We are living at a time when a growing emphasis is laid on increasing the efficiency of management processes, work productivity and optimization of all other activities taking place in a company. Mathematical methods have been used with increasing success in solving particular tasks of dynamic process modeling, both in macroeconomics and microeconomics (optimal process management, dynamic balance models, growth models, etc.). Models describing the processes affected both by the present and history can be described using the so-called ordinary delay differential equations.

The paper presents the possibility of solving a nonlinear dynamic model of production, storage and sale of daily consumer goods, provided that certain parts of the system react with delay. The paper aims to verify the solvability of the described system of ordinary differential equations on the premise of delayed reactions of some variables, and to verify a hypothesis on real data that the length of response to changes influences the system's stabilization. The scientific aim is to verify the solvability of this task. Methods of analysis and synthesis and methods of mathematical analysis (methods of solving functional differential equations) were used to meet the aim. In the paper the model solution is illustrated by a practical task together with a graphical interpretation of the solution.

2. LITERATURE REVIEW

A microeconomic mathematical model may help improve the management of a particular production unit. Dynamic models, described by differential equations, represent one of the ways to

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describe the behaviour of a microeconomic system, whose dynamical properties play an important role.

(Zhang et al.,2014) and (Chen and Hu, 2012) model the function of deterministic demand in a supply chain, while monitoring price effects. They work on the assumption of balanced prices and study in detail the profit sensitivity to changes in various factors. Competitive relationships in a supply chain are analysed in (Chung et al., 2014). The robust optimization model in which demand is regarded as a decreasing function is described in (Lim, 2013).

Dynamic pricing models and their potential extensions are discussed e.g. in (Rana and Olivier, 2014). Studies (Gallego and Hu, 2014) address dynamical pricing of perishable goods. In the publication (Baumol and Blinder, 2015) the authors look into the consumer choice theory, and the maximization of utility in the case of budget constraints. (Sauquet et al., 2011) deal with exogenous price elasticity of demand.

The Duopoly model seen from the point of view of the game theory is expounded in the paper (Shulman and Geng, 2013), where they consider two companies and heterogeneous consumers. There are two publications on air transport based on an empirical study. In the work (Brueckner et al., 2015) the authors analysed the effect of luggage fees on flight ticket prices in air transport. (Nicolae et al., 2016) deal with the issue of delayed flights in connection with the number of checked-in luggage.

One of the models emphasised by contemporary scientific studies is a product-inventory model, which covers production as well as the inventory of finished goods. The traditional approach to the solution to this problem, in terms of optimization, is described in (Sethi and Thompson, 2005), while (Ortega and Lin, 2004) focus on the application of management theories in the field of product-inventory models. This field has seen numerous papers published in the past years, such as publication (Teng and Chang, 2005) on the application of models of stock-dependent demand rate. Another sphere of interest is represented by papers on inventory optimization, e.g. (Bakker et al., 2012).

A further example can be (Tokarev, 2002), who proposed a microeconomic model of short-term crediting and debt repayment for a small company, or a model (Grigorieva and Khailov, 2005), which is supplemented with an additional condition.

The model describing the production process and the sale of goods using the system of nonlinear equations is described in the paper (Grigorieva and Khailov, 2005), who based their work on the principles specified in (Lancaster, 2012) when drawing up their model. The papers expound possible solutions from the point of the theory of optimal management. Application of Pontryagin's principle for solving a similar problem is described by Grigorieva and Khailov in the publication (Grigorieva and Khailov,2015). Papers demonstrate nonlinear mathematical models of a microeconomic system which produces and sells fast moving consumer goods; this paper compares the model with a statistical model.

3. PROBLEM OUTLINE

Let us have a nonlinear mathematical model of a microeconomic system which deals with the production, warehousing and sale of consumer goods. The model is based on works by (Grigorieva and Khailov, 2005), and (Gorsky and Lokshin, 2002), which describe its various variants in detail and discuss solutions to such models. The model can be described by a nonlinear system of differential equations:

$$\begin{gathered} x_1'(t) = u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) - k_1 x_1(t) \\ x_2'(t) = n(p)\big(Y - x_2(t)\big)x_1(t) + k_2 x_2(\mathrm{t}) \\ x_1(0) = x_1^0 \geq 0, x_2(0) = x_2^0 \geq 0, t \in [0, T] \end{gathered} \tag{1}$$

x_1- Number of goods on the market,
x_2- Customers' reserves,
$u(t)$- Rate of production,
k_1 - Rate of goods damage ($k_1 x_1(t)$ became defective per time unit),
k_2 - Consumption rate,
p - Selling price,
n - Coefficient of the rate of sale of perishable goods $n(p) = n_0/p$, $n_0$is an invariable,
Y - Potential demand.

When examining the dependence of some economic quantities in time, we need to take into consideration the fact that a quantity is dependent on its previous values, or on previous values of other quantities. In other words, the effects of time delay in this process must be allowed for.
Let us now admit that changes caused by gradual consumption on the customer's part and damage on the vendor's part may manifest themselves with a certain delay, which means that the system is affected by past changes. If such changes are admitted and feedback is added to the system, a new system will appear:

$$x_1'(t) = u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) + k_1 x_1(t - \Delta_1)$$
$$x_2'(t) = n(p)\big(Y - x_2(t)\big)x_1(t) + k_2 x_2(t - \Delta_2) \qquad (2)$$
$$x_1(0) = x_1^0 \geq 0,\ x_2(0) = x_2^0 \geq 0, t \in [0, T],$$

where Δ_1 and Δ_2 express the time needed to detect changes in variables $x_1$and x_2, and they will still be called delays.
Given the existing knowledge of solving differential equations with delayed argument (more generally, with deviated argument), or so-called functional differential equations, we can apply methods of solution of this particular mathematical model, thus achieving a wider range of results, including a comparison of effects caused by various parameters.

4. SOLVABILITY OF THE NEW SYSTEM

Let us now analyze the above mentioned problem using the modern theory of so-called functional differential equations, a very special part of which is also the theory of linear difference equations with delayed argument. In order to numerically solve the problem in question for the system of differential equations with delayed argument, a method was used, which has been, in current studies, derived for solving marginal problems for systems of so-called functional differential equations – see Bobalová and Maňásek (2007).
General theory, which makes it possible to solve not only the above mentioned problems, but also others, can be found in a monograph Kiguradze and Půža (2003), and its application on the above mentioned types of differential equations with delay, including the description of the way the desired solution was designed can be found in for example Bobalová and Maňásek (2007) and the bibliography cited therein.
For the solution of the mathematical model describing the economic process described in the introduction at the interval $[-\Delta, T]$, the system must be modified into the form:

$$'(t) = u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) + k_1\Big(\chi(t - \Delta_1)x_1(t - \Delta_1) + \big(1 - \chi(t - \Delta_1)\big)x_{1_h}(t - \Delta_1)\Big)$$
$$x_2'(t) = n(p)\big(Y - x_2(t)\big)x_1(t) + k_2\Big(\chi(t - \Delta_2)x_2(t - \Delta_1) + \big(1 - \chi(t - \Delta_2)\big)x_{2_h}(t - \Delta_2)\Big)(3)$$
$$x_1(0) = x_{1h}(0) \geq 0, x_2(0) = x_{2h}(0) \geq 0, t \in [0, T],$$

Where

$$\chi(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases},$$

and x_{ih}, $i = 1{,}2$ are continuous functions at the interval $[-\Delta_i, 0]$.

After applying modern theories of solving differential equations with delayed argument, we created an algorithm that enables finding a solution to the model. The calculations of the solution to our problem were performed using Maple. The functions implemented in Maple cover a wide area of mathematics from the fundamentals of linear algebra, differential and integral count, through differential equations, geometry to logic. Its indisputable advantage is the possibility of finding a solution, even symbolically in some cases. For the solution of our model, the numerical processes aimed for solving ordinary differential equations were used, as a part of Maple.

5. ILLUSTRATIVE EXAMPLE – PLASTIC CUP PRODUCTION

The following example is applied on data acquired in a manufacturing company supplying plastic cups to companies engaged in the vending industry. I.e. the company is a supplier in vending machine sale and operation, in particular coffee vending machines, which is a specific service sector. With regard to the situation in the market, the company has to cope with the high pressure on price reductions while the operating costs are increasing at the same time. Moreover, there is constant competition in this area, which requires consistent efforts to optimize processes.

Now let us work with the assumption that the cups are supplied to vending machines placed at busy locations, therefore they must be refilled several times a week. The result is the need to react promptly to the supply requirements. This paper presents a model reflecting the market situation and examines the effect of historical function change and product sale price, i.e. one of the model parameters.

The first case is based on the information on cup sale available at time t=0. The number of items is given in thousands for better readability. Goods demand $y = 1100$. The amount of goods on the market $x_1 = 850$ and the degree of damage to the goods in storage is $k_1 = 0{,}2$. The production level is maintained at level $u = 650$. The historical functions were based on a particular development that was monitored in the course of 5 weeks and their functions were estimated as follows level of goods on the market $x_1(t) = 850 + 15 \cdot sin(0.5t)$ and level of customer stock$x_2 = 300 + 20 \cdot sin(4t)$. Degree of consumption $k_2 = 0{,}85$. Delay of market response to the changes is defined as $\Delta = 3$ and the delay of customer response is $\Delta = 2$. The sale price of goods is $p = 0{,}2$ monetary units, sale level coefficient $n = 14 \cdot 10^{-5}$. The time unit is one week, the solution is considered for the interval of $t =< 0{,}30 >$.

In the event of a stable market situation when there is no change in the parameters, the amount of goods on the market as well as the level of permanent stock of customers is stabilized (Figure 1).

However, this situation is artificial as changes are happening constantly in the real world. Therefore it is necessary to take account of situations when some of the model parameters are not constant but change continuously. Moreover, the changes may be expected not to occur gradually and continuously, but rather as step changes.

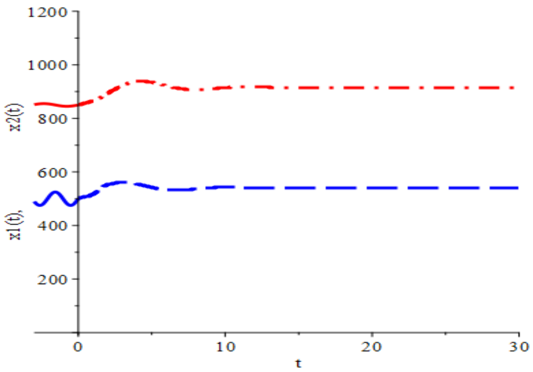

Figure 1. Basic Setting Model
Source: own calculation

The following graphs show a situation when the price is not considered a constant long-term and it changes constantly.

a) t <= 8 p=0.1,
8 < t and t <= 15 p=0.15
t>15 p= 0.2

b) t <= 8, p=0.15,
8 < t and t <= 15, p=0.2,
t>15 p=0.1

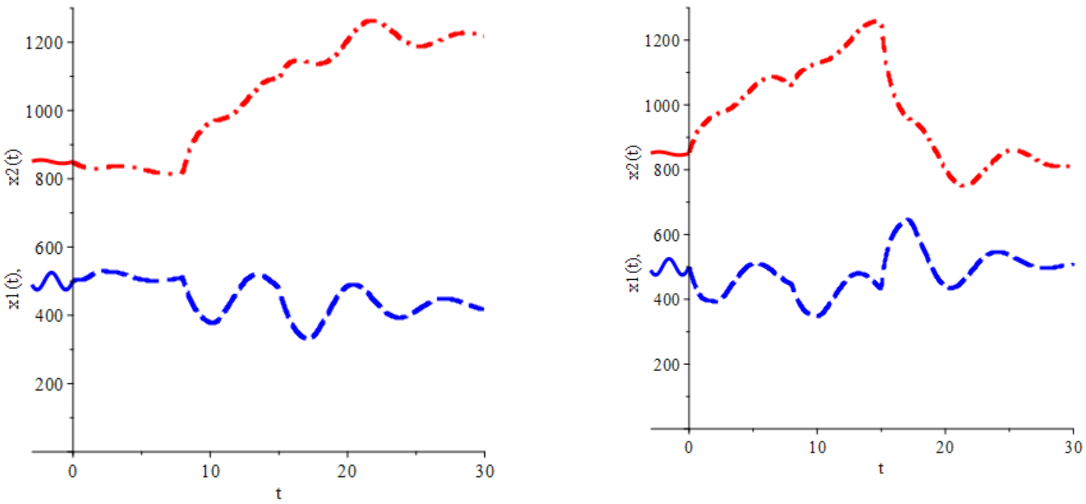

Figure 2. "Step" price change
*Source:*own calculation

The graphs illustrating the situations when step changes occur are shown in Graph 2. In the first case (Graph 2a) the price increased gradually. In this case it is obvious that the customer stock fluctuates; nevertheless the amplitude is decreasing and in a longer period of time it would probably stabilize. The stock on the market has increased considerably, which confirms the manufacturer reacts sensitively to the sale price change. In the case when the price returned to its original level after the increase, there is a gradual adjustment of both customer stock level and the amount of goods on the market.

Let us emphasize this model is the initial task for the system of nonlinear differential equations with a constant delay Δ, in parts by constant coefficients and in parts by continuous non-homogeneity. Therefore its solution will be a continuous function, in parts continuously differentiable at the

interval $[0, T]$, and so in this case the numerical process of the solution construction can be applied as well.
The above method of solving delay differential equations enables the simulation of other situations, too.
Now, we can to verify a hypothesis on real data that the length of response to changes influences the system's stabilization.
Let us use the basic assumption of a situation when step changes in price occur, as shown in Graph 2b. And let us add the assumption that the reactions of customers and manufacturers are delayed, but even this delay is variable. Figure 3a shows that the level of customer stock and the amount of goods on the market gradually stabilize, and the fluctuation is even smaller than with a constant delay.
If we admit certain fluctuation of the expected demand level, we can expect gradual stabilization again; however, the changes in status will happen faster and the fluctuation will be smaller (see Graph 3b).

a) Variable delay:
$d1 = 2 + 0.8\cos(0.3t), d2 = 1 + 0.7\sin(3.3t)$

b) Variable demand:
$\mathrm{Y} = 1100 + 50\cos(3\mathrm{t})$

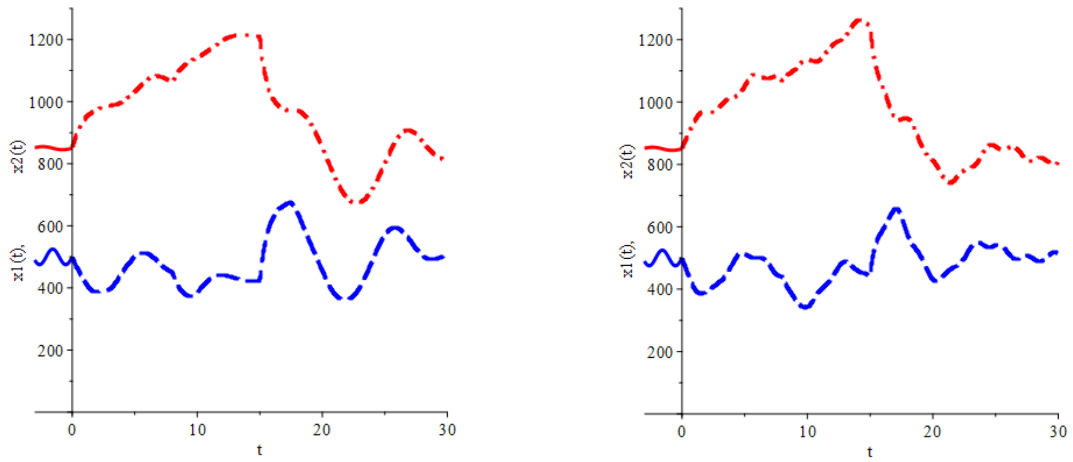

Figure 3. Change in other parameters
*Source:*own calculation

6. CONCLUSION

When modelling complex economic problems, we often face the reality that the relationships between the individual values vary in time. If we consider time to be a continuous value even if the measurements are performed only at certain times, dynamic models can be described using differential equations. The dynamic nature of the model can then be captured by including the delayed effect of the variables. Models describing the processes affected both by the present and history can be described using the so-called ordinary delay differential equations.
This paper presents the possible solution of a model represented by a set of nonlinear differential equations with delayed argument. The set is solved using modern theory and the effect of selected parameter change on the solution is monitored. The illustrative example presents the particular results to the reader graphically as well. To conclude it can be stated that the examined system definitely demonstrates complex dynamic behaviour. Based on the analysis of the above solutions to delay differential equations one can state that a change in the system parameters has a considerable impact on general system stabilization. If we are able to estimate future changes in

parameters, conclusions can be drawn, concerning future behaviour of the system and stabilization possibilities.

To conclude it can be stated that the presented way of solving a system of nonlinear equations can contribute significantly to solving particular tasks based on the data of a real company, enabling the compilation of a production plan and pricing strategy in order to maximize profit. Our solution also enables the degree of impact of change in various factors to be assessed and thus facilitates the selection of an optimal management strategy.

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